CHARACTERIZATION OF γ-FACTORS: THE ASAI CASE

GUY HENNIART AND LUIS LOMELÍ

ABSTRACT. Let E be a separable quadratic algebra over a locally compact field F of positive characteristic. The Langlands-Shahidi method can be used to define the Asai γ-factors for a smooth irreducible generic representation π of $\mathrm{GL}_n(E)$. If σ is the Weil-Deligne representation of $\mathcal{W}_E$ corresponding to π under the local Langlands correspondence, then it is shown that the Asai γ-factor is the same as the γ-factor on the Galois side corresponding to the representation of $\mathcal{W}_E$ obtained from σ under tensor induction. This is achieved by proving that Asai γ-factors are characterized by their local properties together with their role in global functional equations for L-functions. An immediate application concerns the stability of γ-factors under twists by highly ramified characters.

1. INTRODUCTION

Let F be a locally compact field of positive characteristic p. Let ψ be a non-trivial character F and π a smooth irreducible generic representation of $\mathrm{GL}_n(F)$, where n is a positive integer. If ρ_n denotes the standard representation of $\mathrm{GL}_n(\mathbb{C})$, let r be either $\mathrm{Sym}^2\,\rho_n$ or $\wedge^2\rho_n$. In [7] the authors establish the equality of γ-factors:

$$\gamma(s,\pi,r,\psi)=\gamma(s,r\circ\sigma,\psi),$$

where the factor on the left is defined via the Langlands-Shahidi method [14, 15], and σ on the right is the the Weil-Deligne representation corresponding to π under the local Langlands correspondence [13]. The same question in characteristic zero remains open, although much progress has been made [6].

In this paper, we address the case of Asai γ-factors and related L- and ε-factors. These factors can be seen as a generalization of those studied in [1] by T. Asai. Let E/F be a separable quadratic extension of locally compact fields of positive characteristic and let $\overline{F}$ be a separable algebraic closure containing E. Let π be a smooth irreducible representation of $\mathrm{GL}_n(E)$. The L-group of $\mathrm{Res}_{E/F}\,\mathrm{GL}_n$ is $\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})\rtimes\mathcal{W}_F$, where the Weil group $\mathcal{W}_F$ acts via the Galois group $\mathrm{Gal}(E/F)=\{1,\theta\}$. The Asai representation $r_{\mathcal{A}}=r_{\mathcal{A}_n}$ can be defined by

$$\begin{aligned}&r_{\mathcal{A}}:\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})\rtimes\mathrm{Gal}(E/F)\to\mathrm{GL}_{n^2}(\mathbb{C}),\\&r_{\mathcal{A}}(x,y,1)=(x\otimes y)\text{ and }r_{\mathcal{A}}(x,y,\theta)=(y\otimes x).\end{aligned}$$

The Langlands-Shahidi method is used in [15] to define Asai γ-factors $\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi)$ in characteristic p; we rely on that construction in the current paper. Writing σ as

Date: October 2009.
Key words and phrases. Langlands correspondence, local factors, Langlands-Shahidi method.
2000 *Mathematics Subject Classification.* Primary 22E50, 11F70.
The second author was partially supported by the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.

the Weil-Deligne representation of $\mathcal{W}_E$ corresponding to π under local Langlands, we prove that

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi)=\gamma_F^{\mathrm{Gal}}(s,\mathrm{I}(\sigma),\psi), \tag{1.1}$$

where $\mathrm{I}(\sigma)$ denotes the representation of $\mathcal{W}_F$ obtained from σ by tensor induction (see Theorem 3.3). In the case of characteristic zero, equation (1.1) for $n=2$ is known [11, 17] (see [6] for progress in the general case).

Theorem 3.3 is proved via a characterization of Asai γ-factors involving local properties together with their connection with the global theory by means of a functional equation described below (1.2). More precisely, the local properties of $\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi)$ include: a naturality property with respect to isomorphisms of quadratic extensions E/F; an isomorphism property pertaining to π; a dependence on the additive character ψ, which can be made explicit; a crucial multiplicativity property with respect to parabolic induction, which reflects the influence of taking tensor induction on a direct sum of Weil-Deligne representations; and finally, (1.1) is needed whenever the representation π is the generic component of an unramified principal series.

Let K/k be a quadratic separable extension of global function fields of characteristic p, for a split place v of K we have $K\otimes k_v\cong k_v\times k_v$. Thus, in the local theory, the case of a separable quadratic algebra E/F is treated simultaneously. The connection with the global theory is now given by the global functional equation:

> Let Ψ be a non-trivial character of $\mathbb{A}_k/k$. Given a place v of k, let $K_v=K\otimes k_v$. Let S be a finite set of places such that K/k, Π and Ψ are unramified outside of S. Then
>
> $$L^S(s,\Pi,r_{\mathcal{A}})=\prod_{v\in S}\gamma_{K_v/k_v}(s,\Pi_v,r_{\mathcal{A}},\Psi_v)\,L^S(1-s,\check{\Pi},r_{\mathcal{A}}), \tag{1.2}$$
>
> where
>
> $$L^S(s,\Pi,r_{\mathcal{A}})=\prod_{v\notin S}L(s,\Pi_v,r_{\mathcal{A}}).$$

In the course of proving our main results, we directly establish a local-to-global argument for the case of a cuspidal, tamely ramified representation π of $\mathrm{GL}_n(E)$ (hence π of level zero) via the Grundwald-Wang theorem. Then the general problem is reduced to the case of a tamely ramified representation π. This is done by using a local-to-global result due to Gabber and Katz for ℓ-adic representations of the Galois group [10], and translating it via the global Langlands correspondence [12]. We note that care must be taken, since we are considering a quadratic extension E/F.

In the Langlands-Shahidi method, π is assumed to be generic. But, using the Langlands-Zelevinsky classification together with multiplicativity, the definition of γ-factors can be extended to the general case (see § 4.1). Also, we show that the local L- and ε-factors are the same as the corresponding Galois factors. In § 4.2, we take the opportunity to write down a stability property of γ-factors that is not known in characteristic zero. Finally, in § 5 we give a short proof of the equality of local factors studied in [19] for Rankin-Selberg products of GL_m and GL_n.

The second author kindly thanks F. Shahidi for his help and encouragement; he also thanks M. Krishnamurthy and P. Kutzko for useful discussions.

2. Asai γ-factors and tensor induction

2.1. Fix a prime number p and consider the class $\mathscr{L}_{\text{quad.}}(p)$ of triples $(E/F, \pi, \psi)$ consisting of:

- F a locally compact field of characteristic p;
- E a separable quadratic algebra over F, i.e., either E/F is a separable quadratic extension of local fields or $E \simeq F \times F$.
- π a smooth irreducible representation of $\mathrm{GL}_n(E)$, $n \geq 1$;
- ψ a non-trivial character of F.

Given a triple $(E/F, \pi, \psi) \in \mathscr{L}_{\text{quad.}}(p)$, we say it is of degree n if π is a representation of $\mathrm{GL}_n(E)$. Let $x \mapsto \bar{x}$ denote conjugation in E/F, i.e., the non-trivial automorphism of E/F. Let q be the cardinality of its residue field and let $\mathfrak{p}$ be its maximal ideal. Given a representation σ, let $\check{\sigma}$ denote its contragredient and let $\sigma^{\text{conj.}}$ denote the representation obtained from σ by conjugation.

Let $\mathbf{G} = \mathrm{U}(2n)$ be the quasi-split unitary group with respect to E/F. This group can be obtained via the hermitian form

$$h(x, y) = \sum_{i=1}^{2n} \bar{x}_i y_{2n+1-i}.$$

Asai γ-factors in characteristic p are defined in [15] via the Langlands-Shahidi method. They arise from generic representations π of $M = \mathbf{M}(F)$, where $\mathbf{M} = \mathrm{Res}_{E/F} \mathrm{GL}_n$ is the Siegel Levi subgroup of $\mathbf{G}$ (for general π we refer to § 4.1 below). Asai γ-factors give a rule which, to a triple $(E/F, \pi, \psi) \in \mathscr{L}_{\text{quad.}}(p)$, associates a rational function $\gamma_{E/F}(s, \pi, r_{\mathcal{A}}, \psi)$ of $\mathbb{C}(q^{-s})$.

2.2. Given a representation σ of $\mathcal{W}_E$, we consider $\mathrm{I}(\sigma)$ to be tensor induction from $\mathcal{W}_E$ to $\mathcal{W}_F$. (See § 13 of [3]). Given $(E/F, \psi, \pi) \in \mathscr{L}_{\text{quad.}}(p)$, let $\sigma = \sigma(\pi)$ be the representation of $\mathcal{W}_E$ corresponding to π via the local Langlands correspondence. The Galois γ-factors arising in connection with the Asai γ-factors will be written:

$$\gamma_F^{\mathrm{Gal}}(s, \mathrm{I}(\sigma), \psi).$$

These factors satisfy a number of easily established properties, including a multiplicativity property reflecting the decomposition rule:

$$\mathrm{I}(\sigma \oplus \tau) \simeq \mathrm{I}(\sigma) \oplus \mathrm{I}(\tau) \oplus \mathrm{Ind}_E^F(\sigma \otimes \tau^{\text{conj.}}).$$

The Asai γ-factors satisfy the corresponding properties, which we list in the next section.

3. Characterization of Asai factors

3.1. We first give the local properties of Asai γ-factors:

(i) (Naturality). *Let $(E/F, \psi, \pi) \in \mathscr{L}_{\text{quad.}}(p)$ be of degree n, and let η be an isomorphism $\eta : E'/F' \simeq E/F$. Then $\psi' = \psi \circ \eta|_F$ is a non-trivial additive character of F'. Also, via η, π defines a smooth irreducible generic representation π' of $\mathrm{GL}_n(E')$. Then*

$$\gamma_{E/F}(s, \pi, r_{\mathcal{A}}, \psi) = \gamma_{E'/F'}(s, \pi', r_{\mathcal{A}}, \psi').$$

(ii) (Isomorphism). *Let* $(E/F,\psi,\pi) \in \mathscr{L}_{\mathrm{quad.}}(p)$ *be of degree* n*, and let* π' *be a smooth irreducible generic representation of* $\mathrm{GL}_n(E)$ *isomorphic to* π*. Then*

$$\gamma_{E/F}(s,\pi',r_{\mathcal{A}},\psi) = \gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi).$$

For the relationship with Artin factors, see § 5 of [15]. In the case $n = 1$, π can be viewed as a character χ of $E^\times$. Then $\gamma_{E/F}(s,\chi,r_{\mathcal{A}},\psi)$ is equal to the abelian γ-factor $\gamma_F(s,\chi|_{F^\times},\psi)$.

(iii) (Relation with Artin factors). *Let* $(E/F,\psi,\pi) \in \mathscr{L}_{\mathrm{quad.}}(p)$ *be of degree* n*, and assume that* $n = 1$ *or that* π *is the generic component of a principal series. Let* $\sigma = \sigma(\pi)$ *be the Weil-Deligne representation of* $\mathcal{W}_E$ *associated to* π *via the local Langlands correspondence. Let* $\mathrm{I}(\sigma)$ *be the representation of* $\mathcal{W}_F$ *obtained from* σ *by tensor induction. Then*

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi) = \gamma_F^{\mathrm{Gal}}(s,\mathrm{I}(\sigma),\psi).$$

We write this as the next property of Asai γ-factors.

(iv) (Dependence on ψ). *Let* $(E/F,\psi,\pi) \in \mathscr{L}_{\mathrm{quad.}}(p)$ *be of degree* n*, and let* $a \in F^\times$*. Then* $\psi^a : x \mapsto \psi(ax)$ *is a non-trivial additive character of* F *and we have*

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi^a) = \omega_\pi(a)^n |a|_F^{n^2(s-\frac{1}{2})} \gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi).$$

Let us give a short proof of (iv), it relies on the definition given in § 5.1 of [15], which we refer to for any unexplained notation. Consider π as a representation of $M \simeq \mathrm{GL}_n(E)$ and assume it is χ_0-generic, where χ_0 is obtained from ψ. Let

$$t = \mathrm{diag}(a^{-(n-1)}\alpha, a^{-(n-2)}\alpha, \ldots, a\alpha, \alpha, \beta, a^{-1}\beta, \ldots, a^{n-2}\beta, a^{n-1}\beta) \in T(E),$$

where $\alpha\beta^{-1} = \bar{\alpha}^{-1}\bar{\beta} = a \in F^\times$. Then $w_0(t)t^{-1}$ lies in the center of M. Let π_t be given by $\pi_t(x) = \pi(t^{-1}xt)$. The character $\chi_{0,t}$ given by $\chi_{0,t}(u) = \chi_0(t^{-1}ut)$ is then obtained from ψ^a and π_t is $\chi_{0,t}$ generic. Using the definition and a direct computation we get

$$\begin{aligned}\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi^a) &= \lambda(\psi,w_0)\, C_{\overline{\chi}_{0,t}}(s,\breve{\pi}_t,w_0) \\ &= \omega_{\breve{\pi}}(a)^n |a|_F^{n^2(s-\frac{1}{2})}\, \lambda(\psi,w_0)\, C_{\overline{\chi}_0}(s,\breve{\pi},w_0) \\ &= \omega_\pi(a)^n |a|_F^{n^2(s-\frac{1}{2})}\, \gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi).\end{aligned}$$

The following can be found in § 5 of [15]:

(v) (Multiplicativity of γ-factors). *For* $i = 1,\ldots,d$*, let* $(E/F,\psi,\pi_i) \in \mathscr{L}_{\mathrm{quad.}}(p)$ *be of degree* n_i*. Let* $n = n_1 + \cdots + n_d$ *and let* π *be the unique generic component of the representation of* $\mathrm{GL}_n(E)$ *parabolically induced from* $\pi_1 \otimes \cdots \otimes \pi_d$*. Then*

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi) = \prod_{i=1}^{d} \gamma_{E/F}(s,\pi_i,r_{\mathcal{A}_{n_i}},\psi) \prod_{i<j} \gamma_E(s,\pi_i \times \pi_j^{\mathrm{conj.}},\psi \circ \mathrm{Tr}_{E/F}).$$

(Here, each $\gamma_E(s,\pi_i \times \pi_j^{\mathrm{conj.}},\psi \circ \mathrm{Tr}_{E/F})$ *is a Rankin-Selberg factor; see § 5).*

Notice that (v) and the case $n = 1$ give (iii).

(vi) (Split case). *Let* $(E/F,\psi,\pi) \in \mathscr{L}_{\text{quad.}}(p)$ *be of degree* n *and assume* $E \simeq F \times F$. *Then* $\pi \simeq \pi_1 \otimes \pi_2$, *where* π_1 *and* π_2 *are smooth irreducible representations of* $\mathrm{GL}_n(F)$ *and*

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi) = \gamma_F(s,\pi_1 \times \pi_2,\psi),$$

(Again, $\gamma_F(s,\pi_1 \times \pi_2,\psi)$ *is a Rankin-Selberg factor; see § 5).*

3.2. The link between the local and global theory is provided by the following property (see § 5 of [15]):

(vii) (Global functional equation). *Let* K/k *be a quadratic separable extension of global function fields of characteristic* p. *Let* Ψ *be a non-trivial character of* $\mathbb{A}_k/k$ *and let* $\Pi = \otimes'\Pi_v$ *be an automorphic cuspidal representation of* $(\mathrm{Res}_{K/k}\,\mathrm{GL}_n)(\mathbb{A}_k) \simeq \mathrm{GL}_n(\mathbb{A}_K)$. *Given a place* v *of* k, *let* $K_v = K \otimes k_v$. *Let* S *be a finite set of places such that* K/k, Π *and* Ψ *are unramified outside of* S. *Then*

$$L^S(s,\Pi,r_{\mathcal{A}}) = \prod_{v\in S} \gamma_{K_v/k_v}(s,\Pi_v,r_{\mathcal{A}},\Psi_v)\, L^S(1-s,\check{\Pi},r_{\mathcal{A}}),$$

where

$$L^S(s,\Pi,r_{\mathcal{A}}) = \prod_{v\notin S} L(s,\Pi_v,r_{\mathcal{A}}).$$

3.3. **Theorem.** *There is only one rule on* $\mathscr{L}_{\text{quad.}}(p)$ *satisfying properties* (i)$-$(vii). *In particular, for* $(E/F,\psi,\pi) \in \mathscr{L}_{\text{quad.}}(p)$ *of degree* n, *we have*

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi) = \gamma_F^{\mathrm{Gal}}(s,\mathrm{I}\,(\sigma)\,,\psi),$$

where $\sigma = \sigma(\pi)$ *is associated to* π *via the local Langlands correspondence. Moreover, Asai* γ*-factors also satisfy:*

(viii) (Twisting by unramified characters). *Let* $(E/F,\psi,\pi) \in \mathscr{L}_{\text{quad.}}(p)$. *Then*

$$\gamma_{E/F}(s,\pi \otimes |\det(\cdot)|_E^{\frac{s_0}{2}},r_{\mathcal{A}},\psi) = \gamma_{E/F}(s+s_0,\pi,r_{\mathcal{A}},\psi).$$

(ix) (Local functional equation).

$$\gamma_{E/F}(s,\pi,r_{\mathcal{A}},\psi)\ \gamma_{E/F}(1-s,\check{\pi},r_{\mathcal{A}},\overline{\psi}) = 1.$$

3.4. *Proof of theorem.* Property (viii) can be shown directly, and the local functional equation is a property of γ_F^{Gal} that can be immediately translated to $\gamma_{E/F}$. To prove the main result, we can assume, by multiplicativity, that π is cuspidal; as the case $E \cong F \times F$ is given by (vi), we may assume E/F is a quadratic extension. We proceed by induction on $n \geq 1$, where $n = 1$ is given by property (iii). Thus, we consider $(E/F,\psi,\pi) \in \mathscr{L}_{\text{quad.}}(p)$ of degree $n > 1$.

Case 1: $n > 1$, E/F tame, π cuspidal and tame. (Thus π is of level zero). Let σ be the corresponding Galois representation under local Langlands. Then σ is irreducible and is given by

$$\sigma = \mathrm{Ind}_{\mathcal{W}_{E'}}^{\mathcal{W}_E}(\chi),$$

where E'/E is an unramified extension of degree n and $\chi : \mathcal{W}_{E'} \to \mathbb{C}^\times$ is a tame character. By class field theory, χ is the same as a character $\chi : E'^\times \to \mathbb{C}^\times$. Moreover, χ restricted to $U_{E'}$ is obtained from a regular character of $\Bbbk_{E'}^\times$. (Notation: given a local field F, its residue field is denoted by $\Bbbk_F$; given a global function field k, its field of constants is denoted by $\Bbbk_k$).

Let $k = \Bbbk_F(t)$ and let K/k be a separable quadratic extension with $K_0/k_0 \simeq E/F$ and such that $\Bbbk_K = \Bbbk_E$. Let $\Bbbk_n$ be a degree n extension of $\Bbbk_K$, then the constant field extension $K' = \Bbbk_n \cdot K$ is a cyclic extension of degree n, unramified everywhere. Then $K'_0/K_0 \simeq E'/E$. Let w be a place of K that splits completely in K'/K. (Notice that a place w of K splits completely in K'/K if n divides the degree of w). Let $S = \{0, w'_1, \ldots, w'_n\}$, where $w'_i | w$. We can now proceed as in § 2.3 of [7] and construct a character

$$\xi : \prod U_{K'_w} \to \mathbb{C}^\times,$$

where the product ranges over all places w' of K', such that:

- $\xi_0 = \chi|_{U_{K'_0}}$;
- $\xi_{w'} = 1$ if $w' \notin S$, and
- $\xi|_{\Bbbk_{K'}^\times} = 1$.

Then ξ further extends to a grössencharacter

$$\tilde{\xi} : K'^\times \backslash \mathbb{A}_{K'}^\times \to \mathbb{C}^\times .$$

After globally twisting by an unramified character, we can assume that $\tilde{\xi}_0 = \chi$. Also, $\tilde{\xi}_{w'}$ will be unramified for $w' \notin S$.

A grössencharacter $\tilde{\xi}$ as above is the same as a character of $\mathcal{W}_{K'}$ via global class field theory. Then

$$R = \mathrm{Ind}_{\mathcal{W}_{K'}}^{\mathcal{W}_K} \tilde{\xi}$$

will have $R_0 = \rho$ and R_v will be reducible for all places v of K, with $v \neq 0$. Indeed, R_v is unramified for $v \notin \{0, w\}$, and R_w is a sum of characters because w is split in K'/K.

Let ℓ, $\ell \neq p$, be a fixed prime number, and let $\iota : \overline{\mathbb{Q}}_\ell \tilde{\to} \mathbb{C}$ be a fixed field isomorphism. Then, R gives rise to a continuous degree n $\overline{\mathbb{Q}}_\ell$-representation Σ of $\mathcal{W}_K$. The global Langlands correspondence, proved in [12], gives a cuspidal automorphic representation $\Pi = \Pi(\Sigma)$. By the local Langlands correspondence of [13], Π_v corresponds to Σ_v.

By construction: π corresponds to Π_0, Π_v is unramified for $v \notin \{0, w\}$, and Π_w is a principal series representation. (If the place u of k lying below w splits in K/k, then we can use property (vi) instead).

By properties (i) and (ii), we can assume $F = K_0$ and $\pi = \Pi_0$. By property (iv), we can also assume $\psi = \Psi_0$. The global functional equation then gives

$$\prod_{v \in S} \gamma_{K_v/k_v}(s, \Pi_v, r_{\mathcal{A}}, \Psi_v) = \prod_{v \in S} \gamma_{k_v}^{\mathrm{Gal}}(s, \mathrm{I}(\Pi_v), \Psi_v), \tag{3.1}$$

where S is a finite set of places of k containing 0 and u, $w|u$. But, for v distinct from 0 and u, Π_v is an unramified principal series. Also, Π_u is a (possibly ramified) principal series. Hence, property (vii) (and property (vi) if u splits) gives

$$\gamma_{K_v/k_v}(s, \Pi_v, r_{\mathcal{A}}, \Psi_v) = \gamma_{k_v}^{\mathrm{Gal}}(s, \mathrm{I}(\Pi_v), \Psi_v), \ v \in S - \{0\}\,. \tag{3.2}$$

Then (3.1) and (3.2) give equality at 0.

Case 2. $n > 1$, E/F general and π cuspidal, but not necessarily of level zero. Let $\sigma = \sigma(\pi)$ be the corresponding irreducible Weil-Deligne representation. Also, twisting by an unramified character if necessary, we can assume σ is a representation of the Galois group. Then σ factors through some Galois group $\mathrm{Gal}(E'/E)$. Let $\tilde{E}$ be the Galois closure of E'/F. Consider $k = \Bbbk_F(t)$ and find a Galois extension $\tilde{K}$

of k, such that: $\tilde{K}$ is $\tilde{E}$ at 0, tame at ∞ and unramified elsewhere. This is possible by the results of Gabber-Katz [10]; moreover, notice that $\mathrm{Gal}(\tilde{E}/F)$ is the same as $\mathrm{Gal}(\tilde{K}/k)$ in Katz's construction. Then, if we let K/k be the quadratic extension corresponding to E/F, σ gives a representation Σ of $\mathrm{Gal}(\tilde{K}/k)$. The representation Σ will be σ at 0, tame at ∞ and unramified elsewhere.

By the global Langlands correspondence of [12], there exists an irreducible cuspidal automorphic automorphic representation $\Pi = \Pi(\Sigma)$. The local components Π_v of Π are obtained from Σ_v via the the local Langlands correspondence of [13]. Since we assumed σ to be a representation of the Galois group, we may need to twist globally by an unramified character to ensure that $\Pi_0 = \pi$, but this does not affect the properties of Π. Then we can apply (3.1) and (3.2) to Π, with $S = \{0, \infty\}$, in order to complete the proof. The equality at ∞ is given by CASE 1 treated above and by the fact that if Σ_∞ is tame, then so are all its irreducible components. □

4. LOCAL L-FUNCTIONS, ROOT NUMBERS AND STABILITY

4.1. **Equality of local factors.** Let us recall the definition of local L-functions and ε-factors via the Langlands-Shahidi method. Since the local factors we are studying arise from GL_n, they can be defined for $(E/F, \psi, \pi_i) \in \mathscr{L}_{\mathrm{quad.}}(p)$ where π is any smooth irreducible representation of $\mathrm{GL}_n(E)$. We note that the following discussion can be used to extend the results of [7], related to $\wedge^2 \rho_n$ and $\mathrm{Sym}^2 \rho_n$, to representations that are not necessarily generic.

Let $(E/F, \psi, \pi) \in \mathscr{L}_{\mathrm{quad.}}(p)$. Let us first assume that π is tempered, then π is generic [22]. Let $P_\pi(t)$ be the unique polynomial satisfying $P_\pi(0) = 1$ and such that $P_\pi(q^{-s})$ is the numerator of $\gamma_{E/F}(s, \pi, r_{\mathcal{A}}, \psi)$. Then

$$L(s, \pi, r_{\mathcal{A}}) := P_\pi(q^{-s})^{-1}.$$

Because π is tempered, $L(s, \pi, r_{\mathcal{A}})$ is holomorphic for $\mathrm{Re}(s) > 0$. If π is parabolically induced from $\pi_1 \otimes \cdots \otimes \pi_d$, where each π_i is tempered, then multiplicativity of γ-factors gives multiplicativity for the L-functions:

$$L(s, \pi, r_{\mathcal{A}}) = \prod_{i=1}^{d} L(s, \pi_i, r_{\mathcal{A}_{n_i}}) \prod_{i<j} L(s, \pi_i \times \pi_j^{\mathrm{conj.}}).$$

The local ε-factor is defined to satisfy the relation:

$$\gamma_{E/F}(s, \pi, r_{\mathcal{A}}, \psi) = \varepsilon_{E/F}(s, \pi, r_{\mathcal{A}}, \psi) \frac{L(1-s, \tilde{\pi}, r_{\mathcal{A}})}{L(s, \pi, r_{\mathcal{A}})}. \tag{4.1}$$

Given $(E/F, \psi, \pi) \in \mathscr{L}_{\mathrm{quad.}}(p)$ in general, we can use Langlands classification to write π as parabolically induced from $\pi_{\nu,1} \otimes \cdots \otimes \pi_{\nu,d}$, where each $\pi_{\nu,i}$ is quasi-tempered with a negative Langlands parameter ν. Each $\pi_{0,i}$ is tempered, and the L-functions $L(s, \pi_{\nu,i}, r_{\mathcal{A}})$ and $L(s, \pi_{\nu,i} \times \pi_{\nu,j})$ are defined by analytic continuation on ν. Then

$$L(s, \pi, r_{\mathcal{A}}) := \prod_{i=1}^{d} L(s, \pi_{\nu,i}, r_{\mathcal{A}_{n_i}}) \prod_{i<j} L(s, \pi_{\nu,i} \times \pi_{\nu,j}^{\mathrm{conj.}}),$$

and the root numbers are defined to satisfy (4.1).

This is in accordance with the way local L-functions and ε-factors are defined for Weil-Deligne representations. Equality of local factors follows first for tempered representations from Theorem 3.3. Then in general by the above discussion.

4.2. **Stability of γ-factors.** Let us briefly recall the stability property of local factors for Weil-Deligne representations [4, 5]. Let σ be a Weil-Deligne representation. Let η be a character of $F^\times$ of level k, for k sufficiently large (depending on σ). Take an element $c = c(\eta, \psi) \in F^\times$ such that $\psi(cx) = \eta(1+x)$ for $x \in \mathfrak{p}^{[k/2]+1}$. Then

$$\varepsilon(s, \sigma \otimes \eta, \psi) = \det(\sigma(c))^{-1} \varepsilon(s, \eta, \psi)^{\dim \sigma},$$
$$L(s, \sigma \otimes \eta) = L(s, \check{\sigma} \otimes \eta^{-1}) = 1.$$

Because of this, the next property is now a corollary to Theorem 3.3. We phrase it in terms of γ-factors.

Corollary (Stability). *Let $(E/F, \psi, \pi_i) \in \mathscr{L}_{\text{quad.}}(p)$, $i = 1, 2$, both of the same degree. Assume that π_1 and π_2 have the same central character. Then, for every sufficiently highly ramified character η of $F^\times$, we have*

$$\gamma_{E/F}(s, \eta \cdot \pi_1, r_{\mathcal{A}}, \psi) = \gamma_{E/F}(s, \eta \cdot \pi_2, r_{\mathcal{A}}, \psi).$$

Remark 1. *It is clear that the same result holds for local factors corresponding to exterior and symmetric square L-functions. To be more precise, we use the notation of* [7]*: Let $(F, \psi, \pi_i) \in \mathscr{L}(p)$, $i = 1, 2$, both of degree n. Assume that π_1 and π_2 have the same central character. Then, for every sufficiently highly ramified character η of $F^\times$, we have*

$$\gamma_F(s, \eta \cdot \pi_1, r_n, \psi) = \gamma_F(s, \eta \cdot \pi_2, r_n, \psi).$$

Notice that, by the discussion in § 4.1, the representations π_i need not be generic.

5. Rankin selberg products for representations of GL_m and GL_n

5.1. Let $(F, \psi, \pi_i) \in \mathscr{L}_{\text{quad.}}(p)$, $i = 1$, 2. The Rankin-Selberg γ-factor

$$\gamma(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \pi_1 \times \pi_2, \psi) \, \frac{L(1-s, \check{\pi}_1 \times \check{\pi}_2)}{L(s, \pi_1 \times \pi_2)}$$

is defined in [8]. Consider $\mathbf{M} = \mathrm{GL}_m \times \mathrm{GL}_n$ as a maximal Levi subgroup of $\mathbf{G} = \mathrm{GL}_{m+n}$ and let $\mathbf{P} = \mathbf{MN}$ be the maximal standard parabolic subgroup with Levi $\mathbf{M}$ and unipotent radical $\mathbf{N}$. The adjoint action of ${}^L M$ on ${}^L\mathfrak{n}$ is $r \simeq \rho_m \otimes \check{\rho}_n$. For this r, the γ-factors $\gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi)$ are defined in [14, 15] via the Langlands-Shahidi method. The aim of [19] is to establish the equality

$$\gamma(s, \pi_1 \times \pi_2, \psi) = \gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi) \tag{5.1}$$

using completely local methods. What we now provide is a short proof of this result by means of a characterization of γ-factors.

5.2. **Proof of equation** (5.1)**.** Let $\mathbf{B} = \mathbf{TU}$, be the Borel subgroup of GL_{m+n} consisting of upper triangular matrices. Let χ_0 be the character of $\mathbf{U}(F)$ obtained from ψ and, abusing notation, we also write χ_0 for the restriction of χ_0 to $U_M = \mathbf{M}(F) \cap \mathbf{U}(F)$; they will be w_0-compatible in the notation of [15]. Consider $\sigma = \pi_1 \otimes \check{\pi}_2$ as a representation of the Levi M. We may assume σ is χ_0-generic.

Both $\gamma(s, \pi_1 \times \pi_2, \psi)$ and $\gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi)$ satisfy naturality and isomorphism properties. The multiplicativity property of the local coefficient implies multiplicativity for $\gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi)$. For $\gamma(s, \pi_1 \times \pi_2, \psi)$, multiplicativity can be found in Theorem 3.1 of [8]. The relation with Artin factors when π_1 and π_2 are principal series is reduced via multiplicativity to establishing the relation in the case of GL_2, which is well known.

For $a \in F^\times$, let $t = \mathrm{diag}(a^{-(m+n-1)}, a^{-(m+n-2)}, \ldots, a, 1)$. Let $\sigma = \pi_1 \otimes \check{\pi}_2$ and let σ_t be given by $\sigma_t(x) = \sigma(t^{-1}xt)$. The character $\chi_{0,t}$ given by $\chi_{0,t}(u) = \chi_0(t^{-1}ut)$ is then obtained from ψ^a and σ_t is $\chi_{0,t}$ generic. Using the definition and a direct computation to compare both local coefficients we obtain:

$$\begin{aligned}\gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi^a) &= C_{\overline{\chi}_{0,t}}(s, \check{\sigma}_t, w_0) \\ &= \omega_{\check{\pi}_1}(a)^{-m}\omega_{\pi_2}(a)^n |a|_F^{mn(s-\frac{1}{2})}\, C_{\overline{\chi}_0}(s, \check{\sigma}, w_0) \\ &= \omega_{\pi_1}(a)^m\omega_{\pi_2}(a)^n |a|_F^{mn(s-\frac{1}{2})}\, \gamma(s, \pi_1 \otimes \check{\pi}_2, r, \psi).\end{aligned}$$

The same relationship holds for $\gamma(s, \pi_1 \times \pi_2, \psi)$.

Finally, we have a global functional equation: Let K be a global function field of characteristic p, let $\Psi = \otimes_v \Psi_v$ be a non-trivial character of $K\backslash\mathbb{A}_K$, and let Π_1 and Π_2 be cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A}_K)$ and $\mathrm{GL}_n(\mathbb{A}_K)$, respectively. Let S be a finite set of places of K such that Ψ and Π_i, for $i = 1, 2$, are unramified outside of S. Then, Theorem 5.14 of [14] gives

$$L^S(s, \Pi_1 \times \Pi_2) = \prod_{v \in S} \gamma(s, \Pi_{1,v} \otimes \check{\Pi}_{2,v}, r, \Psi_v)\, L^S(1 - s, \check{\Pi}_1 \times \check{\Pi}_2).$$

The functional equation for $\gamma(s, \Pi_1 \times \Pi_2, \psi)$ can be found in [2, 16].

Given local representations π_1 and π_2, we use the local-global argument in the proof of Theorem 3.3 to prove (5.1). A brief outline should suffice: by multiplicativity, assume π_1 and π_2 are cuspidal. Use Proposition 2.2 of [7] to deal with the case when π_1 and π_2 are both cuspidal of level zero. Then, use Proposition 3.1 of [7] for general cuspidal representations.

Remark 2. *The above argument should also hold in characteristic zero by using Proposition 5.1 of* [20] *as the link between the local and the global theory, relying on the global functional equation* [2, 18]. *The theory for archimedean local fields is studied in* [9, 21].

Remark 3. *The local properties of $\gamma(s, \pi_1 \times \pi_2, \psi)$ used above can also be obtained via the local Langlands correspondence in any characteristic.*

References

[1] T. Asai, *On certain Dirichlet series associated with Hilbert modular forms and Rankin's method*, Math. Ann. **226** (1977), 81-94.

[2] J. Cogdell, *Notes on L-functions for* GL_n, ICTP Lecture Notes, 2000.

[3] C. W. Curtis and I. Reiner, *Methods of representation theory I*, John Wiley & Sons, New York, 1981.

[4] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable II, LNM **349**, Springer Verlag, 1973.

[5] P. Deligne and G. Henniart, *Sur la variation, par torsion, des constantes locales d'équations fonctionnelles des fonctions L*, Invent. Math. **64** (1981), 89-118.

[6] G. Henniart, *Correspondance de Langlands et fonctions L des carrés extérieur et symétrique*, preprint.

[7] G. Henniart and L. Lomelí, *Local-to-global extensions for* GL_n *in non-zero characteristic: a characterization of* $\gamma(s, \pi, \mathrm{Sym}^2, \psi_F)$ *and* $\gamma(s, \pi, \wedge^2, \psi_F)$, preprint.

[8] H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika, *Rankin-Selberg convolutions*, Am. J. Math. **105** (1983), 367-464.

[9] H. Jacquet and J. Shalika, *Rankin-Selberg convolutions: archimedean theory*, in *Festschrift in Honor of I.I. Piatetski-Shapiro*, Part I, Weizmann Science Press, 1990, 125-207.

[10] N. M. Katz, *Local-to-global extensions of representations of fundamental groups*, Ann. Inst. Fourier **36** (1986), 69-106.

[11] M. Krishnamurthy, *The Asai transfer to* GL_4 *via the Langlands-Shahidi method*, IMRN **2003**, no. 41 (2003), 2221-2254.

[12] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), 1-241.

[13] G. Laumon, M. Rapoport and U. Stuhler, *$\mathcal{D}$-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), 217-338.

[14] L. Lomelí, *Functoriality for the classical groups over function fields*, International Mathematics Research Notices 2009; Vol. 2009: article ID rnp089, 65 pages, doi:10.1093/imrn/rnp089

[15] L. Lomelí, *On the Langlands-Shahidi Method for the classical groups in non-zero characteristic and applications*, preprint.

[16] I. I. Piatetski-Shapiro, *Zeta functions for* $\mathrm{GL}(n)$, preprint of the University of Maryland, 1976.

[17] D. Ramakrishnan, *Modularity of solvable Artin representations of $GO(4)$-type*, IMRN **2002**, no. 1 (2002), 1-54.

[18] F. Shahidi, *On certain L-Functions*, Amer. J. Math. **106** (1981), 297-355.

[19] F. Shahidi, *Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$*, Amer. J. Math. **106** (1984), 67-111.

[20] ———, *A proof of Langlands' conjecture on Plancherel measures; complementary series of $\mathfrak{p}$-adic groups*, Ann. Math. **132** (1990), 273-330.

[21] ———, Local coefficients as Artin factors for real groups, Duke Math. J. **52** (1985), 973-1007.

[22] A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups II. On irreducible representations of GL_n*, Ann. scient. Éc. Norm. Sup., 4^e série, **13** (1980), 165-210.

Guy Henniart, Univ. Paris-Sud, Laboratoire de Mathématiques d'Orsay, CNRS, Orsay cedex F-91405, France

E-mail address: `Guy.Henniart@math.u-psud.fr`

Luis Lomelí, Department of Mathematics, The University of Iowa, 15 MacLean Hall, Iowa City, IA 52242, USA

E-mail address: `llomeli@math.uiowa.edu`